Simultaneous Equations and Instrumental Variables

Nirav Mehta
(Based on Roy Allen's materials)

Instrumental Variables

We study a new tool that can be used to estimate parameters of economic models in three cases where OLS can fail.

Common theme: in the regression we would otherwise run, the regressor is correlated with the unobservables, so the zero conditional mean (SLR.4/MLR.4) condition fails.

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Some economic models describe how *several* variables are determined.

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Supply Curve:

$$Q_S = \tilde{\gamma}_0 + \tilde{\gamma}_1 P + \varepsilon_S.$$

Demand Curve:

$$Q_D = \tilde{\delta}_0 + \tilde{\delta}_1 P + \varepsilon_D.$$

Markets Clear:

$$Q_S = Q_D.$$

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Suppose observation i is generated according to the model. In other words, these equations hold

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⚠ Regressing Q on P will not consistently estimate $\tilde{\gamma}_1$ or $\tilde{\delta}_1$.

- We will analyze why and develop a new tool to estimate the supply and demand curves.

Simultaneous Equations

Set supply equal to demand,

$$\tilde{\gamma}_0 + \tilde{\gamma}_1 P_i + u_{i,S} = \tilde{\delta}_0 + \tilde{\delta}_1 P_i + u_{i,D}.$$

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$$P_i = \frac{\tilde{\delta}_0 - \tilde{\gamma}_0}{\tilde{\gamma}_1 - \tilde{\delta}_1} + \frac{u_{i,D} - u_{i,S}}{\tilde{\gamma}_1 - \tilde{\delta}_1}$$

⚠ P_i and the unobservables are necessarily correlated.

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⚠ P_i and the unobservables are necessarily correlated.

- With this setup, P_i is actually a function of the unobservables (it is endogenous).
- We use some additional **exogenous** variables that shift the supply or demand curve.
- By *exogenous*, we mean not systematically related to the unobservables. In the IV setup, we will formalize this with covariance restrictions.

Simultaneous Equations

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$$Q_i = \tilde{\delta}_0 + \tilde{\delta}_1 P_i + u_{i,D}$$

- We will show how a regression can be used to estimate $\tilde{\delta}_1$, i.e. the slope of the demand curve.
- This requires several steps and a lot of algebra (sorry).

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Set supply equal to demand

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We can solve for P_i to get

$$\begin{aligned} P_i &= \frac{1}{\tilde{\gamma}_1 - \tilde{\delta}_1} \left[\tilde{\delta}_0 - \tilde{\gamma}_0 - \tilde{\gamma}_2 Z_{i,1} + u_{i,D} - u_{i,S} \right] \\ &= \left(\frac{\tilde{\delta}_0 - \tilde{\gamma}_0}{\tilde{\gamma}_1 - \tilde{\delta}_1} \right) - \left(\frac{\tilde{\gamma}_2}{\tilde{\gamma}_1 - \tilde{\delta}_1} \right) Z_{i,1} + \left(\frac{u_{i,D} - u_{i,S}}{\tilde{\gamma}_1 - \tilde{\delta}_1} \right). \end{aligned}$$

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We can rewrite this as...

$$P_i = \beta_0 + \beta_1 Z_{i,1} + e_i.$$

Simultaneous Equations

Key equations:

$$P_i = \left(\frac{\tilde{\delta}_0 - \tilde{\gamma}_0}{\tilde{\gamma}_1 - \tilde{\delta}_1}\right) - \left(\frac{\tilde{\gamma}_2}{\tilde{\gamma}_1 - \tilde{\delta}_1}\right) Z_{i,1} + \left(\frac{u_{i,D} - u_{i,S}}{\tilde{\gamma}_1 - \tilde{\delta}_1}\right)$$
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- We can learn *something* about the parameters of the supply and demand system!

⚠ We *need* $\tilde{\gamma}_2 \neq 0$. The shifter Z_1 needs to actually shift the supply curve.

Simultaneous Equations

We can plug this equation for P_i back into the demand equation to motivate a different regression.

$$
\begin{aligned}
Q_i &= \tilde{\delta}_0 + \tilde{\delta}_1 \frac{1}{\tilde{\gamma}_1 - \tilde{\delta}_1} \left[\tilde{\delta}_0 - \tilde{\gamma}_0 - \tilde{\gamma}_2 Z_{i,1} + u_{i,D} - u_{i,S} \right] + u_{i,D} \\
&= \alpha_0 + \alpha_1 Z_{i,1} + r_i.
\end{aligned}
$$

We can estimate α_1 by regressing Q on Z_1 if we assume $\mathbb{E}[r_i \mid Z_1] = 0$.

- An estimate of α_1 gives us an estimate of $\left(\frac{-\tilde{\delta}_1 \tilde{\gamma}_2}{\tilde{\gamma}_1 - \tilde{\delta}_1} \right)$
- We can learn something else about the parameters of the supply and demand system!

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We can estimate $\tilde{\delta}_1$, the slope of the demand curve.

- We can estimate it by the *ratio* of two regression coefficients...
- $\hat{\tilde{\delta}}_1 = \frac{\hat{\alpha}_1}{\hat{\beta}_1}$
  - $\hat{\beta}_1$ was from a regression of P on Z_1.
  - $\hat{\alpha}_1$ was from a regression of Q on Z_1.

Simultaneous Equations

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Given a supply shifter (Z_1), we can estimate the slope of the demand curve in a two-step process.

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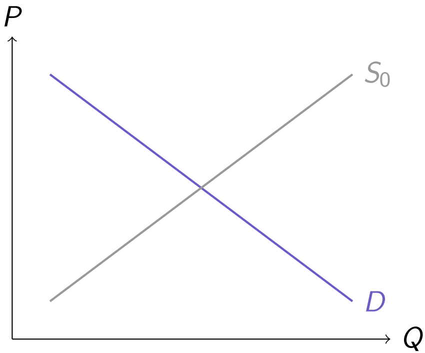
This technique is a form of *instrumental variables* (IV).

Simultaneous Equations

The presence of Z_1 (a supply shifter) was key to estimating the demand curve.

The same technique applies if we have a demand shifter. We can then estimate the supply curve.

Simultaneous Equations: Graphical Intuition



Supply: $Q_S = \tilde{\gamma}_0 + \tilde{\gamma}_1 P + \tilde{\gamma}_2 Z_1 + \varepsilon_S$

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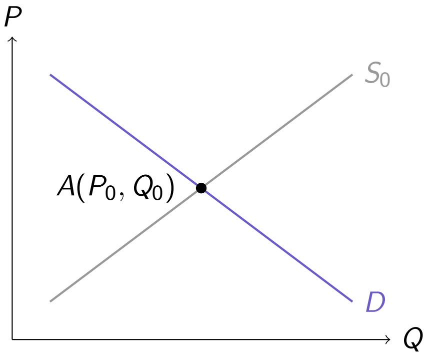

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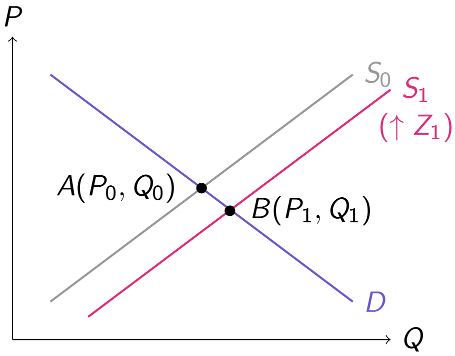

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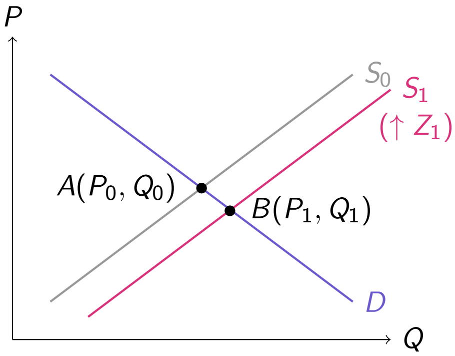

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First reg.: $P_i = \beta_0 + \beta_1 Z_{i,1} + e_i$

Second reg.: $Q_i = \alpha_0 + \alpha_1 Z_{i,1} + r_i$

Slope of demand: $\tilde{\delta}_1 = \alpha_1 / \beta_1$

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Basic idea: prices and quantities were jointly determined, but we can use an exogenous shifter (Z_1) to estimate parameters of the original economic model.

Z_1 is an example of an instrument. We can use such variables to address endogeneity or omitted-variable problems.

Instrumental Variables

As usual, begin with the economic model so we're clear about what we're interested in:

$$Y = \tilde{\beta}_0 + \tilde{\beta}_1 X + \varepsilon.$$

We are interested in $\tilde{\beta}_1$. This tells us how Y changes if X changes, *all else equal*.

Instrumental Variables

The data are generated according to

$$Y_i = \beta_0 + \beta_1 X_i + u_i.$$

We maintain that $\tilde{\beta}_1 = \beta_1$, *but* do not assume that $\mathbb{E}[u \mid X] = 0$.

- This is a violation of SLR.4, so a regression of Y on X may not produce a consistent estimator of $\tilde{\beta}_1$.
- The issue is that X may be systematically related to the unobservables.
- This can happen because of omitted variables, simultaneity, or measurement error.

Instrumental Variables

We assume we observe an instrument Z_i that has these important properties:

- Exogeneity: $\text{Cov}(Z, u) = 0$.
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We need that Z is *not* systematically related to the unobservables, but *is* related to X itself.

We call Z an *instrument* for X.

In the supply/demand example, the supply shifter was an instrument for price.

Instrumental Variables

Given these assumptions, we use data on Y_i, X_i, Z_i to construct an *instrumental variables* (IV) estimator for $\tilde{\beta}_1$.

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$$\hat{\beta}_1^{IV} = \frac{\sum_{i=1}^{n}(Z_i - \overline{Z})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(Z_i - \overline{Z})(X_i - \overline{X})}$$

⚠ Does this look familiar at all?

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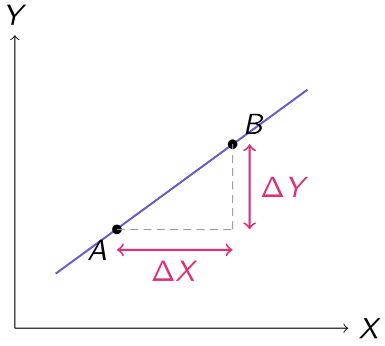
⚠ Does this look familiar at all?

- *In this simple setup*, we can obtain $\hat{\beta}_1^{IV}$ by two regressions.
- First regress X on Z, then regress Y on Z. Take the ratio of the regression coefficients.
- We did this in the price/quantity example earlier.

Instrumental Variables: Rise Over Run



When Z changes from z_0 to z_1, X changes by ΔX, and that induced change in X moves Y by ΔY.

$$\Delta Z = z_1 - z_0$$

- "First stage": $\Delta X / \Delta Z$
- "Second stage": $\Delta Y / \Delta Z$

So the IV slope is the rise from the second regression divided by the run from the first stage:

$$\hat{\beta}_1^{IV} \approx \frac{\Delta Y / \Delta Z}{\Delta X / \Delta Z} = \frac{\Delta Y}{\Delta X}.$$

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What is $\hat{\beta}_1^{IV}$ estimating under our assumptions?

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We get

$$\text{Cov}(Z, Y) = \beta_1 \,\text{Cov}(Z, X) + \text{Cov}(Z, u).$$

Rearranging and applying our assumptions,

$$\beta_1 = \frac{\text{Cov}(Z, Y)}{\text{Cov}(Z, X)}.$$

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Since we assumed $\beta_1 = \tilde{\beta}_1$, we obtain

$$\hat{\beta}_1^{IV} \to \frac{\mathrm{Cov}(Z, Y)}{\mathrm{Cov}(Z, X)} = \tilde{\beta}_1.$$

In a large sample, the IV estimator is close to the parameter we want to know, $\tilde{\beta}_1$.

Measurement Error

We will work through another example, in which *measurement error* in a regressor leads to inconsistency of the OLS estimator.

In other words, in a large sample the OLS estimator typically will not be close to the parameter we are interested in.

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We will then show that instrumental variables (IV) can also help address this problem.

Measurement Error

Suppose as before we are interested in an economic model:

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Now we describe how the data are generated:

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and $\beta_0 = \tilde{\beta}_0$, $\beta_1 = \tilde{\beta}_1$, $\mathbb{E}[u \mid X^*] = 0$.

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What do we get when we regress Y on X? How does it relate to $\tilde{\beta}_1$?

- We will show that, typically, the OLS estimator does *not* consistently estimate $\tilde{\beta}_1$.
- We will also show how instrumental variables (IV) can be used to consistently estimate $\tilde{\beta}_1$.

Measurement Error

To study what the regression of Y on X is estimating, we write,

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_i^* + u_i \\ &= \beta_0 + \beta_1 (X_i - e_i) + u_i \\ &= \beta_0 + \beta_1 X_i + (-\beta_1 e_i + u_i) \\ &= \beta_0 + \beta_1 X_i + u_i'. \end{aligned}$$

What is a tool we can use to understand what $\hat{\beta}_1^{OLS}$ is estimating here?

- Earlier we provided different characterizations of *what* $\hat{\beta}_1^{OLS}$ is estimating in a large sample. Is there a useful one for this?

Measurement Error

In a large sample, the OLS estimator is consistently estimating

$$\hat{\beta}_1^{OLS} \rightarrow \tilde{\beta}_1 + \frac{\text{Cov}(X, u')}{\text{Var}(X)}$$

We can use this to understand the bias of the OLS estimator.

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Remember

$$X_i = X_i^* + e_i$$
$$u_i' = -\tilde{\beta}_1 e_i + u_i$$

Measurement Error

$$
\begin{aligned}
\text{Cov}(X, u') &= \text{Cov}(X^* + e, -\tilde{\beta}_1 e + u) \\
&= \text{Cov}(X^*, -\tilde{\beta}_1 e + u) + \text{Cov}(e, -\tilde{\beta}_1 e + u) \\
&= \left[\text{Cov}(X^*, -\tilde{\beta}_1 e) + \text{Cov}(X^*, u)\right] \\
&\quad + \left[\text{Cov}(e, -\tilde{\beta}_1 e) + \text{Cov}(e, u)\right].
\end{aligned}
$$

Measurement Error

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\end{aligned}
$$

We can plausibly assume three of these terms are 0:

- $\text{Cov}(X^*, e) = 0$: The measurement error (e_i) is not systematically related to X_i^*. This implies $\text{Cov}(X^*, -\tilde{\beta}_1 e) = 0$.
- $\text{Cov}(X^*, u) = 0$: X^* itself is exogenous. (u was the unobservable in the ideal equation $Y_i = \beta_0 + \beta_1 X_i^* + u_i$.)
- $\text{Cov}(e, u) = 0$. The measurement error is uncorrelated with the unobservables u.

Measurement Error

With these assumptions,

$$\text{Cov}(X, u') = \text{Cov}(e, -\tilde{\beta}_1 e) = -\tilde{\beta}_1 \text{Var}(e).$$

- Whenever there is measurement error ($\text{Var}(e) \neq 0$) and X^* actually has a causal effect on Y ($\tilde{\beta}_1 \neq 0$), the OLS estimator is biased and inconsistent.

Measurement Error

In a large sample, the OLS estimator is consistently estimating

$$\hat{\beta}_1^{OLS} \rightarrow \tilde{\beta}_1 + \frac{\text{Cov}(X, u')}{\text{Var}(X)}$$

We can learn more about the bias, since we can also calculate $\text{Var}(X)$.

$$\text{Var}(X) = \text{Var}(X^*) + \text{Var}(e) + 2\,\text{Cov}(X^*, e).$$

We assumed $\text{Cov}(X^*, e) = 0$.

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$$\text{Var}(X) = \text{Var}(X^*) + \text{Var}(e) + 2\,\text{Cov}(X^*, e).$$

We assumed $\text{Cov}(X^*, e) = 0$. So...

$$\hat{\beta}_1^{OLS} \rightarrow \tilde{\beta}_1 + \frac{-\tilde{\beta}_1 \text{Var}(e)}{\text{Var}(X^*) + \text{Var}(e)}$$

Measurement Error

$$\hat{\beta}_1^{OLS} \rightarrow \tilde{\beta}_1 + \frac{-\tilde{\beta}_1 \text{Var}(e)}{\text{Var}(X^*) + \text{Var}(e)}$$

We can rearrange,

$$\begin{aligned}\hat{\beta}_1^{OLS} \rightarrow \tilde{\beta}_1 &\left(1 - \frac{\text{Var}(e)}{\text{Var}(X^*) + \text{Var}(e)}\right) \\ &= \tilde{\beta}_1 \left(\frac{\text{Var}(X^*)}{\text{Var}(X^*) + \text{Var}(e)}\right)\end{aligned}$$

⚠ Can we interpret this? Is there a lesson here?

Measurement Error

This setup is an example of a *classical errors-in-variables* (CEV) setup.

- We assumed $\text{Cov}(X^*, e) = 0$ and $\text{Cov}(e, u) = 0$.
- We concluded that the OLS estimator is systematically closer to 0 than $\tilde{\beta}_1$.
- This is called *attenuation bias* – bias towards 0.

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- This is called *attenuation bias* – bias towards 0.

These assumptions may not hold for *all* forms of measurement error...
But we can develop the intuition that "typically" the OLS estimator is biased towards 0 in the presence of measurement error.

Measurement Error

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Under the CEV assumptions, it is biased towards 0.

Instrumental variables (IV) provide a solution.

- We will use an instrument Z that provides a *second* measurement of the original variable X^*.

Measurement Error

Example:

$$Q = \tilde{\beta}_0 + \tilde{\beta}_1 Inc + \varepsilon.$$

- Q is quantity of instant Ramen.
- Inc is income measured in thousands of Canadian dollars.

Measurement Error

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- Q is quantity of instant Ramen.
- Inc is income measured in thousands of Canadian dollars.

We have data on

- Self-reported income ($RInc_i$).

Measurement Error

We specify a statistical model describing how the data are generated:

$$Q_i = \beta_0 + \beta_1 Inc_i + u_i$$

$$\beta_0 = \tilde{\beta}_0 \qquad \beta_1 = \tilde{\beta}_1$$

$$RInc_i = Inc_i + e_i$$

Measurement Error

We specify a statistical model describing how the data are generated:

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$$\beta_0 = \tilde{\beta}_0 \qquad \beta_1 = \tilde{\beta}_1$$
$$RInc_i = Inc_i + e_i$$

So we obtain,

$$\begin{aligned} Q_i &= \beta_0 + \beta_1 RInc_i + (-\beta_1 e_i + u_i) \\ &= \beta_0 + \beta_1 RInc_i + u_i' \end{aligned}$$

As an instrument, suppose we have a second measure of income:

- Employers' reported income ($EInc_i$).

Measurement Error

$$Q_i = \beta_0 + \beta_1 RInc_i + u_i'$$

We assume that $EInc_i$ is related to $RInc_i$ but is not systematically related to unobservable heterogeneity u_i or the measurement error in $RInc_i$, denoted e_i.

Measurement Error

$$Q_i = \beta_0 + \beta_1 RInc_i + u'_i$$

We assume that $EInc_i$ is related to $RInc_i$ but is not systematically related to unobservable heterogeneity u_i or the measurement error in $RInc_i$, denoted e_i.

Formally,

- Exogeneity: $\text{Cov}(EInc, u') = 0$.
  - Here, this follows if $\text{Cov}(EInc, -\beta_1 e) = 0$ and $\text{Cov}(EInc, u) = 0$.
- $\text{Cov}(EInc, RInc) \neq 0$.
  - We assume these are two measurements of the same thing.

Measurement Error

We can use *EInc* as an instrument for *RInc*. The IV estimator is:

$$\hat{\beta}_1^{IV} = \frac{\sum_{i=1}^{n}(Q_i - \overline{Q})(EInc_i - \overline{EInc})}{\sum_{i=1}^{n}(RInc_i - \overline{RInc})(EInc_i - \overline{EInc})}$$

- As before, this can be calculated as the ratio of two regression coefficients.

Endogeneity

We showed IV can be used with simultaneous equations or measurement error.

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As a last case, we will consider an example with omitted variables.

$$wage = \tilde{\beta}_0 + \tilde{\beta}_1 educ + \varepsilon.$$

Endogeneity

The data are generated according to

$$wage_i = \beta_0 + \beta_1 educ_i + u_i$$
$$\beta_0 = \tilde{\beta}_0 \qquad \beta_1 = \tilde{\beta}_1$$
$$u_i = \gamma_1 ability_i + u_i'$$

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We showed in the omitted variable bias example a few weeks ago that even if $\text{Cov}(educ, u') = 0$,

$$\hat{\beta}_1^{OLS} \to \tilde{\beta}_1 + \frac{\gamma_1 \,\text{Cov}(educ, ability)}{\text{Var}(educ)}.$$

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We showed in the omitted variable bias example a few weeks ago that even if $\text{Cov}(educ, u') = 0$,

$$\hat{\beta}_1^{OLS} \to \tilde{\beta}_1 + \frac{\gamma_1 \, \text{Cov}(educ, ability)}{\text{Var}(educ)}.$$

The OLS estimator may not be estimating what we want ($\tilde{\beta}_1$).

Endogeneity

One famous instrument used to address this sort of problem is proximity to school:

- $nearc4$: whether someone grew up near a four-year college.

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- *nearc*4: whether someone grew up near a four-year college.

Idea: proximity to a university will make it **more likely** someone will go to university, but may plausibly be **unrelated** to ability.

Endogeneity

One famous instrument used to address this sort of problem is proximity to school:

- $nearc4$: whether someone grew up near a four-year college.

Idea: proximity to a university will make it **more likely** someone will go to university, but may plausibly be **unrelated** to ability.

- Relevance: We think $\text{Cov}(educ, nearc4) > 0$.
- Exogeneity: We are willing to assume $\text{Cov}(nearc4, u) = 0$.
  - This is true if $nearc4$ is uncorrelated with ability and other unobservables u'.

General IV

We have discussed a simple instrumental variables setup with one endogenous variable X and one exogenous instrument Z.

IV also works with a more general setup.

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IV also works with a more general setup.

$$\begin{aligned} Y_i = &\beta_0 + \beta_1 X_{1,i} + \cdots + \beta_K X_{K,i} \\ &+ \gamma_1 W_{1,i} + \cdots + \gamma_L W_{L,i} + u_i. \end{aligned}$$

- We can include exogenous control variables $W_1, \ldots, W_L$. (Similar to when we discussed multiple regression.)
- We can have multiple endogenous regressors $X_1, \ldots, X_K$.
- We can have multiple instruments $Z_1, \ldots, Z_J$.

General IV

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- We need instruments to deal with the fact that the X variables are endogenous.
- For J instruments $Z_{1,i}, \ldots, Z_{J,i}$, these must satisfy the exogeneity conditions
  - $\text{Cov}(Z_j, u) = 0$ for each instrument j.
- We also need to formalize the fact that the control variables (W variables) are exogenous.
  - $\text{Cov}(W_\ell, u) = 0$ for each ℓ.

General IV

$$
\begin{aligned}
Y_i = &\beta_0 + \beta_1 X_{1,i} + \cdots + \beta_K X_{K,i} \\
&+ \gamma_1 W_{1,i} + \cdots + \gamma_L W_{L,i} + u_i.
\end{aligned}
$$

We need **at least** one instrument for each endogenous regressor.

- This is called the “order condition.”

We need a stronger condition that is harder to state.

- It is called the “rank condition,” and requires the additional assumption that a certain matrix is invertible.
- When there are no control variables, one endogenous variable (X), and one instrument (Z), the rank condition is the previous relevance condition

$$
\mathrm{Cov}(X, Z) \neq 0.
$$

General IV

Key takeaways:

- IV works in a more general setup.
- Key assumptions are exogeneity conditions and relevance conditions.
- In the general setup the "relevance" condition is more complicated, and is called the "rank condition."

General IV

In applications, the most common use of IV is

- One instrument.
- One endogenous variable.
- Exogenous control variables (W variables).

General IV

⚠ Why did we often use the assumption $\mathbb{E}[u \mid X] = 0$ (SLR.4, MLR.4) when we introduced regression, but now have covariance assumptions in IV?

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⚠ Why did we often use the assumption $\mathbb{E}[u \mid X] = 0$ (SLR.4, MLR.4) when we introduced regression, but now have covariance assumptions in IV?

- In regression, we often used the stronger zero conditional mean assumption $\mathbb{E}[u \mid X] = 0$.
  - For example, if $\mathbb{E}[u \mid X] = 0$ and $\mathbb{E}[u] = 0$, then $\text{Cov}(X, u) = 0$. (You do not need to prove this.)
- In instrumental variables, some variables are endogenous and some are exogenous.
- So in IV we work with covariance restrictions such as $\text{Cov}(Z, u) = 0$ for instruments and $\text{Cov}(W_\ell, u) = 0$ for exogenous controls.